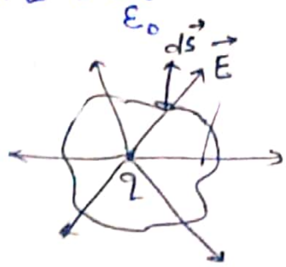
① Electromagnetic wave : Dr. Md. NAIYAR PERWEZ

Derivation of Maxwell's equation for Electromagnetic field

Maxwell's first equation:- $\vec{\nabla}\cdot\vec{D} = \rho$ or $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

From Gauss's law of electrostatics, total electric flux through any closed surface is equal to total charge enclosed by the closed surface divided by ε_0.



i.e, $\oint_S \vec{E}\cdot\vec{ds} = \frac{q}{\varepsilon_0}$ ——— ①

Now volume charge density $\rho = \frac{dq}{dV} \Rightarrow dq = \rho\, dV$

Total charge enclosed in the volume bounded by the closed surface is $q = \int dq \Rightarrow q = \int_V \rho\, dV$ put in eqn ①, we get

$$\oint_S \vec{E}\cdot\vec{ds} = \frac{1}{\varepsilon_0}\cdot\int_V \rho\, dV$$

But from Gauss' divergence theorem, $\int_V (\vec{\nabla}\cdot\vec{E})\, dV = \oint_S \vec{E}\cdot\vec{ds}$

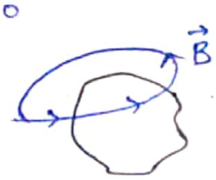
so $\int_V (\vec{\nabla}\cdot\vec{E})\, dV = \frac{1}{\varepsilon_0}\int_V \rho\, dV \Rightarrow \int_V \left(\vec{\nabla}\cdot\vec{E} - \frac{\rho}{\varepsilon_0}\right) dV = 0$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} - \frac{\rho}{\varepsilon_0} = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}} \text{ or } \boxed{\vec{\nabla}\cdot\vec{D} = \rho} \quad \because \vec{D} = \varepsilon_0\vec{E}$$

Physical significance of Maxwell's first equation: Total electric flux density D through the surface enclosing volume is equal to charge density ρ with-in the volume. It means that a charge distribution generates a steady electric field.

Maxwell's second equation:- $\vec{\nabla}\cdot\vec{B} = 0$ or $\vec{\nabla}\cdot\vec{H} = 0$

From Gauss' law of magnetism, total magnetic flux through any closed surface is equal to zero.



i.e, $\oint \vec{B}\cdot\vec{ds} = 0$

Using Gauss' divergence theorem, $\int_V (\vec{\nabla}\cdot\vec{B})\, dV = \oint_S \vec{B}\cdot\vec{ds}$

so $\int_V (\vec{\nabla}\cdot\vec{B})\, dV = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$ or $\boxed{\vec{\nabla}\cdot\vec{H} = 0} \quad \because B = \mu_0 H$

Physical significance of Maxwell's second equation: Net magnetic flux through a closed surface is zero means magnetic monopoles do not exist.

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Maxwell's third equation: $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

From Faraday's law of electromagnetic induction, the induced emf produced in a closed loop is equal to negative of time rate of change in magnetic flux linked with the closed loop.

i.e, induced emf $e = -\frac{\partial \phi}{\partial t}$

Emf (e) is defined as the workdone in carrying unit positive charge (+1C) round a closed circuit.

so $e = \oint_C \vec{E}\cdot d\vec{l} \Rightarrow \oint_C \vec{E}\cdot d\vec{l} = -\frac{\partial \phi}{\partial t}$

But from Stoke's theorem, $\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = \oint_C \vec{E}\cdot d\vec{l}$

so $\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\frac{\delta \phi}{\partial t}$ But $\phi = \int_S \vec{B}\cdot d\vec{s}$

$$\Rightarrow \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\frac{\delta}{\delta t}\int_S \vec{B}\cdot d\vec{s} \Rightarrow \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\int_S \frac{\delta \vec{B}}{\partial t}\cdot d\vec{s}$$

$$\Rightarrow \int_S \left(\vec{\nabla}\times\vec{E} + \frac{\delta \vec{B}}{\partial t}\right)\cdot d\vec{s} = 0 \Rightarrow \vec{\nabla}\times\vec{E} + \frac{\delta \vec{B}}{\partial t} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

physical significance of Maxwell's third equation: Emf around a closed path is equal to time derivative of the magnetic flux density through the surface bounded by the path. It means that an electric field can be generated by time varying magnetic field.

Maxwell's fourth equation:- $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\delta \vec{D}}{\partial t}$

Concept of Displacement current

From Ampere's circuital law, line integral of magnetic field along any closed loop is equal to μ_0 times of total electric current passing through the surface bounded by the closed loop.

i.e, $\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I$ but $I = \int_S \vec{J}\cdot d\vec{s}$ and $\vec{B} = \mu_0\vec{H}$

so $\oint_C \mu_0 \vec{H}\cdot d\vec{l} = \mu_0 \int_S \vec{J}\cdot d\vec{s} \Rightarrow \oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s}$

But from stoke's theorem, $\oint_C \vec{H}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{H})\cdot d\vec{s}$

so $\int_S (\vec{\nabla}\times\vec{H})\cdot d\vec{s} = \int_S \vec{J}\cdot d\vec{s} \Rightarrow \int_S (\vec{\nabla}\times\vec{H} - \vec{J})\cdot d\vec{s} = 0$

$\Rightarrow \vec{\nabla}\times\vec{H} - \vec{J} = 0 \Rightarrow \boxed{\vec{\nabla}\times\vec{H} = \vec{J}}$ ——— (a)

The relation (a) stands only for steady closed current.

Now $\text{div}(\text{curl}\,\vec{H}) = 0$ because divergence of curl of any vector is zero.

$\Rightarrow \text{div}\,\vec{J} = 0$ ——— (b) using equ ⓐ

From equation of continuity, $\text{div}\,J = -\frac{\delta\rho}{\delta t}$ ——— (c)

From equ's ⓑ and ⓒ, $\frac{\delta\rho}{\delta t} = 0 \Rightarrow \rho = \text{constant}$.

Therefore, equ ⓐ is valid only for constant volume charge density. If volume charge density ρ be variable then $\text{div}\,J \neq 0$

Maxwell suggested that equ ⓐ is incomplete for variable volume charge density ρ. Maxwell suggested that use $J + J_d$ in place of J for variable charge density ρ or for variable field in space and Maxwell gave a concept of new quantity J_d known as displacement current density.

Hence equ ⓐ will become $\vec{\nabla}\times\vec{H} = \vec{J} + \vec{J}_d$ ——— (d)

Since $\text{div}(\vec{\nabla}\times\vec{H}) = 0$ so $\text{div}(\vec{J} + \vec{J}_d) = 0$

$\Rightarrow \text{div}\,\vec{J} + \text{div}\,\vec{J}_d = 0 \Rightarrow \text{div}\,\vec{J}_d = -\text{div}\,\vec{J}$

$\Rightarrow \text{div}\,\vec{J}_d = -\left(-\frac{\delta\rho}{\delta t}\right)$ $\because \text{div}\,\vec{J} = -\frac{\delta\rho}{\delta t}$ from equ ⓒ

$\Rightarrow \text{div}\,\vec{J}_d = \frac{\delta\rho}{\delta t} \Rightarrow \text{div}\,\vec{J}_d = \frac{\delta}{\delta t}(\vec{\nabla}\cdot\vec{D})$ $\because$ from Maxwell's first equ. $\vec{\nabla}\cdot\vec{D} = \rho$

$\Rightarrow \vec{\nabla}\cdot\vec{J}_d = \vec{\nabla}\cdot\left(\frac{\delta\vec{D}}{\delta t}\right) \Rightarrow \vec{\nabla}\cdot\left(\vec{J}_d - \frac{\delta\vec{D}}{\delta t}\right) = 0$

$\Rightarrow \vec{J}_d - \frac{\delta\vec{D}}{\delta t} = 0 \Rightarrow \boxed{\vec{J}_d = \frac{\delta\vec{D}}{\delta t}}$ It is formula for displacement current density.

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put $\vec{J}_d = \frac{\delta \vec{D}}{\delta t}$ in equ (d), we get

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t}} \quad \text{or} \quad \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\delta \vec{D}}{\delta t}\right)}$$

physical significance of Maxwell's fourth equation: From Maxwell's fourth equ, magneto motive force around a closed path is equal to sum of conduction current density J and displacement current density $\vec{J}_d = \frac{\delta \vec{D}}{\delta t}$. It means that a magnetic field is generated by time varying electric field.

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* Displacement current density $\vec{J}_d = \frac{\delta \vec{D}}{\delta t}$

The current density produced in a space due to variation of electric flux density $\vec{D}$ with time, is known as displacement current density.

Displacement current produced in space due to variation of electric flux density $\vec{D}$ with time will be

$$I_D = \int_S \vec{J}_d \cdot d\vec{s} = \int_S \frac{\delta \vec{D}}{\delta t} \cdot d\vec{s} = \frac{\delta}{\delta t} \int_S \varepsilon_0 \vec{E} \cdot d\vec{s} \quad \because \vec{D} = \varepsilon_0 \vec{E}$$

$$= \varepsilon_0 \frac{\delta}{\delta t} \cdot \int_S \vec{E} \cdot d\vec{s}$$

$$\Rightarrow \boxed{I_D = \varepsilon_0 \frac{\delta \phi_E}{\delta t}}$$ It is formula for displacement current

Therefore displacement current I_D is produced in space due to variation of electric flux (ϕ_E) with time.

Ques:- Derive (or develop) Maxwell's equation for electromagnetic field and discuss their physical significance.

Ques:- Write short notes on displacement current.